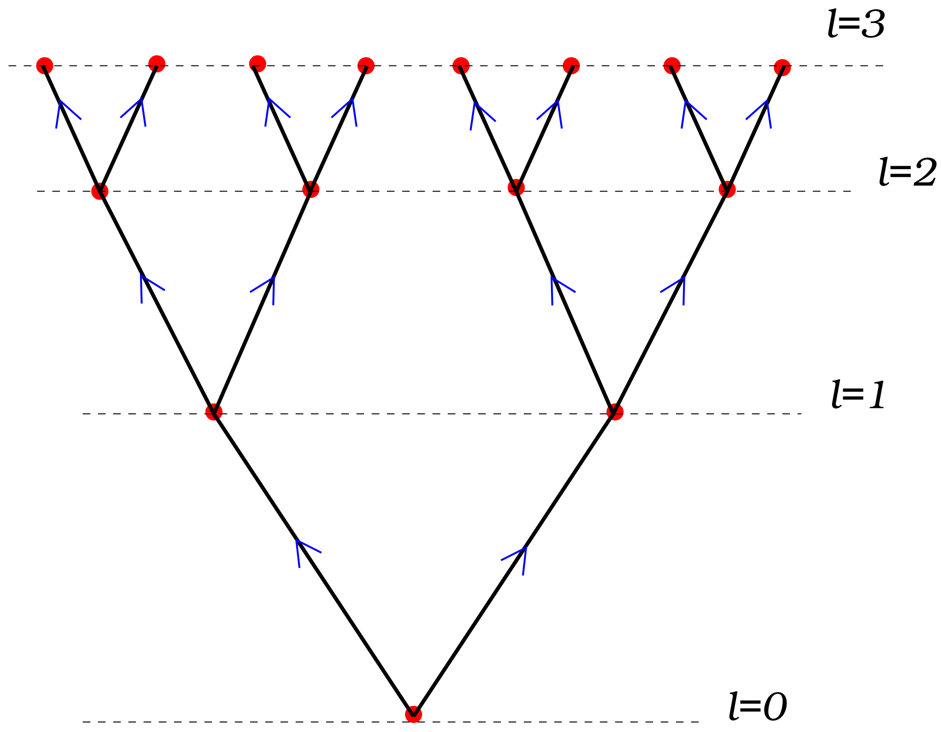
Information broadcast on a tree and reconstruction

Marc Mézard, joint work with Andrea Montanari

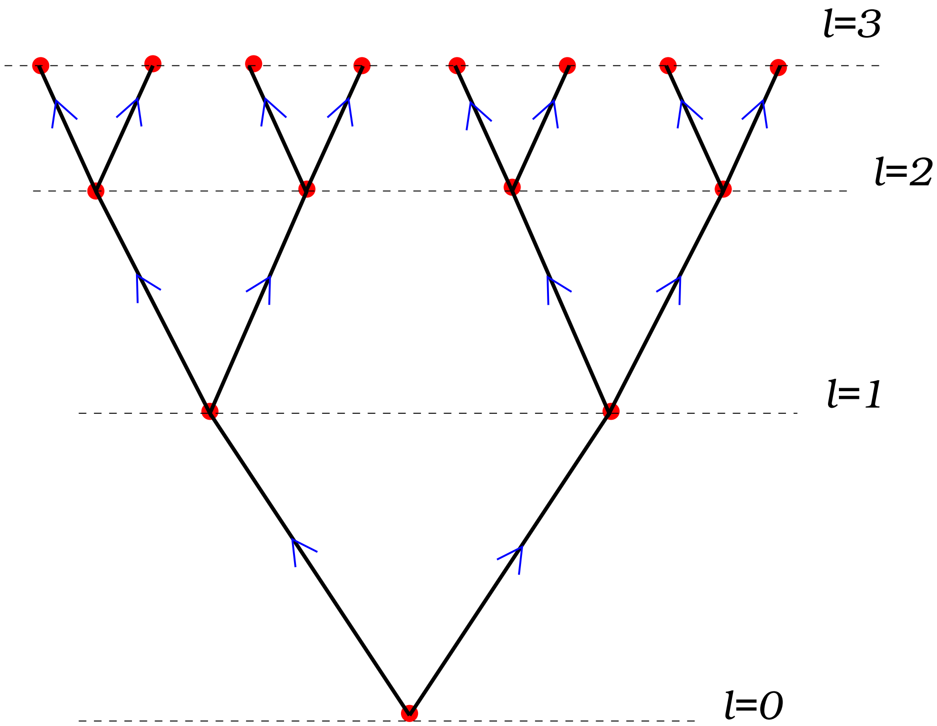
Santa Fe, may 2007

The broadcast/reconstruction problem

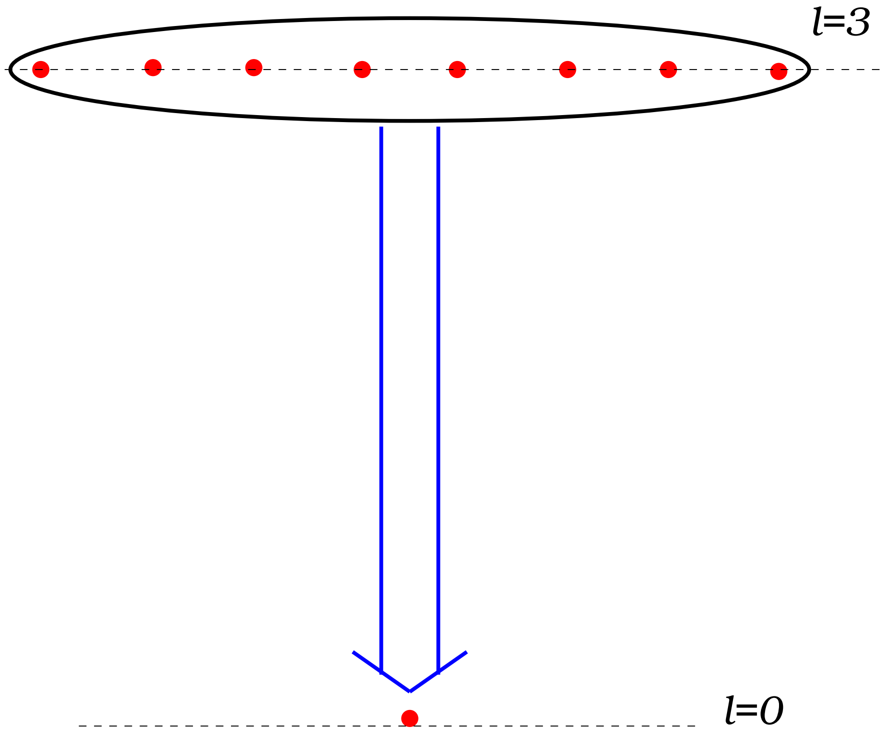


Broadcast on a tree

The broadcast/reconstruction problem



Broadcast on a tree

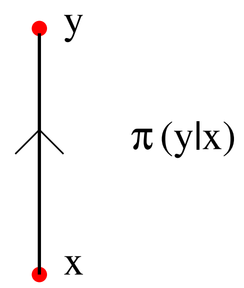


Reconstruction

Motivation

- Communication network
- Propagation of genetic information
- Generalization of Markov chain to trees
- Statistical physics on a Cayley tree / Bethe lattice
- Optimization problems and error correcting codes: locally tree-like networks
- Spin glass phase

Communication channel

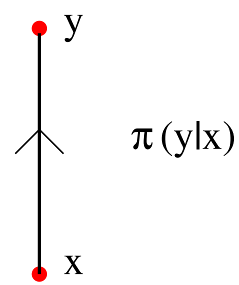


Message from alphabet, e.g. $x, y \in \{1, \ldots, q\}$
Broadcast $x \to y$: probability $\pi(y|x)$.

Example "Ferromagnetic Potts channel":

$$\pi(y|x) = \begin{cases} 1 - \varepsilon & \text{if } y = x \\ \frac{\varepsilon}{q-1} < 1 - \varepsilon & \text{otherwise} \end{cases}$$

Communication channel



Message from alphabet, e.g. $x, y \in \{1, \ldots, q\}$
Broadcast $x \to y$: probability $\pi(y|x)$.

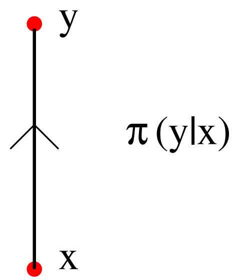
Example "Ferromagnetic Potts channel":

$$\pi(y|x) = \begin{cases} 1-\varepsilon & \text{if } y = x \\ \frac{\varepsilon}{q-1} < 1-\varepsilon & \text{otherwise} \end{cases} = \frac{1}{q-1+e^{\beta}} \exp\left(\beta \delta_{x,y}\right)$$

Noise level in the channel: $T = 1/\beta$ (related to ε by $e^{-\beta} = \frac{\varepsilon}{(q-1)(1-\varepsilon)}$)

$\varepsilon \in [0, \frac{q-1}{q}]$; Larger ε $\to$ Higher temperature.

Communication channel



Message from alphabet, e.g. $x, y \in \{1, \dots, q\}$
Broadcast $x \to y$: probability $\pi(y|x)$.

Example "Ferromagnetic Potts channel":

$$\pi(y|x) = \begin{cases} 1-\varepsilon & \text{if } y = x \\ \frac{\varepsilon}{q-1} < 1-\varepsilon & \text{otherwise} \end{cases} \quad = \frac{1}{q-1+e^{\beta}} \exp\left(\beta\delta_{x,y}\right)$$

Noise level in the channel: $T = 1/\beta$ (related to ε by $e^{-\beta} = \frac{\varepsilon}{(q-1)(1-\varepsilon)}$)

$\varepsilon \in [0, \frac{q-1}{q}]$; Larger $\varepsilon \rightarrow$ Higher temperature.

('Antiferromagnetic' channel: $\varepsilon \in [\frac{q-1}{q}, 1]$. $\varepsilon = 1 \rightarrow$ proper coloring)

Information on the boundary about the root

Broadcast: generates a boundary configuration B.

Reconstruction: Does B contain some information on the letter sent from the root, in the large ℓ limit?

Potts channel broadcasted from $x_0 = 1$:

$\psi_\ell = \sum_B P_{broadcast}(B|x_0 = 1) P(x = 1|B) - \frac{1}{q}$.

Reconstruction possible iff $\lim_{\ell \to \infty} \psi_\ell > 0$.

Phase transition (Mossel):

Rec. possible for $\varepsilon < \varepsilon_r$ (i.e. $T < T_r$), impossible for $\varepsilon > \varepsilon_r$

Reconstruction versus "census reconstruction"

- Single variable on the boundary: correlation with root decays as $e^{-c\ell}$ when $\ell \to \infty$, as soon as $\beta < \infty$.

- Census reconstruction: information contained in the number of boundary sites with $x = 1$?

- Reconstruction: information contained in the full boundary pattern?

A simple upper bound: ferromagnetic transition

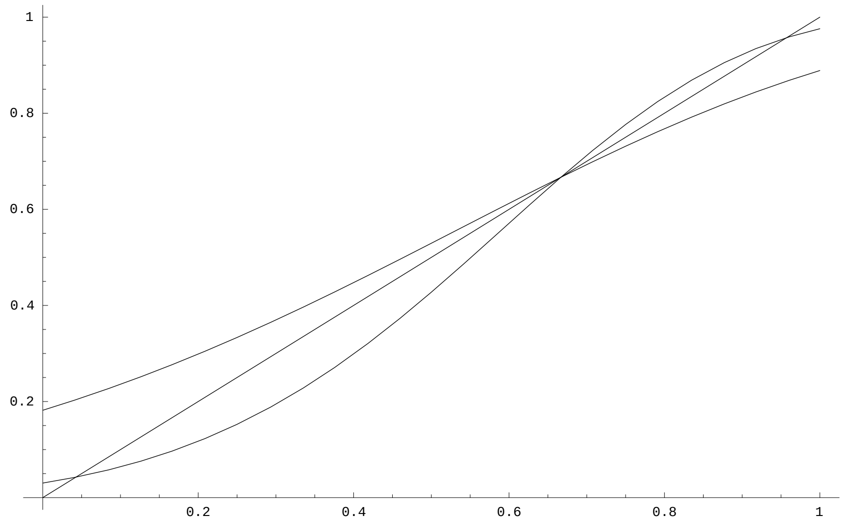
Fully polarized boundary, $x = 1$ on all sites.

Reconstruction. Shell n: probability $\eta^{(n)}(x) = (1-a_n)\delta_{x,1} + \frac{a_n}{q-1}(1-\delta_{x,1})$.

Mapping: $a_{n-1} = F(a_n)$
Boundary condition $a_\ell = 0$.

Fixed point $a = \frac{q-1}{q}$.

Attractive iff $\varepsilon > \varepsilon_F = \frac{q}{q-1}\,\frac{k-1}{k}$



If $T > T_F$ no correlation of center with B $\rightarrow$ reconstruction impossible

Kesten-Stigum lower bound (1966)

Shell n: k^n variables. Assume $x_n\, k^n$ are in state $x = 1$.

$$x_{n+1}\, k^{n+1} = \sum_{i=1}^{x_n\, k^{n+1}} u_i + \sum_{j=1}^{(1-x_n)\, k^{n+1}} z_j$$

$$u_i = \begin{cases} 1 \text{ probability } & 1-\varepsilon \\ 0 \text{ probability } & \varepsilon \end{cases}$$

$$z_j = \begin{cases} 1 \text{ probability } & \frac{\varepsilon}{q-1} \\ 0 \text{ probability } & 1-\frac{\varepsilon}{q-1} \end{cases}$$

Kesten-Stigum lower bound (1966)

Shell n: k^n variables. Assume $x_n\, k^n$ are in state $x = 1$.

$$x_{n+1}\, k^{n+1} = \sum_{i=1}^{x_n\, k^{n+1}} u_i + \sum_{j=1}^{(1-x_n)\, k^{n+1}} z_j$$

$$u_i = \begin{cases} 1 \text{ probability } & 1-\varepsilon \\ 0 \text{ probability } & \varepsilon \end{cases}$$

$$z_j = \begin{cases} 1 \text{ probability } & \frac{\varepsilon}{q-1} \\ 0 \text{ probability } & 1-\frac{\varepsilon}{q-1} \end{cases}$$

Large n: $P(x_n) \sim$ Gaussian

$$\mathbb{E}(x_n) \sim \frac{1}{q} + C \left|1 - \varepsilon\frac{q}{q-1}\right|^n$$

Kesten-Stigum lower bound (1966)

Shell n: k^n variables. Assume $x_n\, k^n$ are in state $x = 1$.

$$x_{n+1}\, k^{n+1} = \sum_{i=1}^{x_n\, k^{n+1}} u_i + \sum_{j=1}^{(1-x_n)\, k^{n+1}} z_j$$

$$u_i = \begin{cases} 1 \text{ probability } & 1-\varepsilon \\ 0 \text{ probability } & \varepsilon \end{cases}$$

$$z_j = \begin{cases} 1 \text{ probability } & \frac{\varepsilon}{q-1} \\ 0 \text{ probability } & 1-\frac{\varepsilon}{q-1} \end{cases}$$

Large n: $P(x_n) \sim$ Gaussian

$$\mathbb{E}(x_n) \sim \frac{1}{q} + C\left|1 - \varepsilon\frac{q}{q-1}\right|^n$$

$$\sqrt{\mathbb{E}({x_n}^2) - [\mathbb{E}(x_n)]^2} \sim C' k^{-n/2}$$

Kesten-Stigum lower bound (1966)

Shell n: k^n variables. Assume $x_n\, k^n$ are in state $x = 1$.

$$x_{n+1}\, k^{n+1} = \sum_{i=1}^{x_n\, k^{n+1}} u_i + \sum_{j=1}^{(1-x_n)\, k^{n+1}} z_j$$

$$u_i = \begin{cases} 1 \text{ probability } & 1-\varepsilon \\ 0 \text{ probability } & \varepsilon \end{cases}$$

$$z_j = \begin{cases} 1 \text{ probability } & \frac{\varepsilon}{q-1} \\ 0 \text{ probability } & 1-\frac{\varepsilon}{q-1} \end{cases}$$

Large n: $P(x_n) \sim$ Gaussian

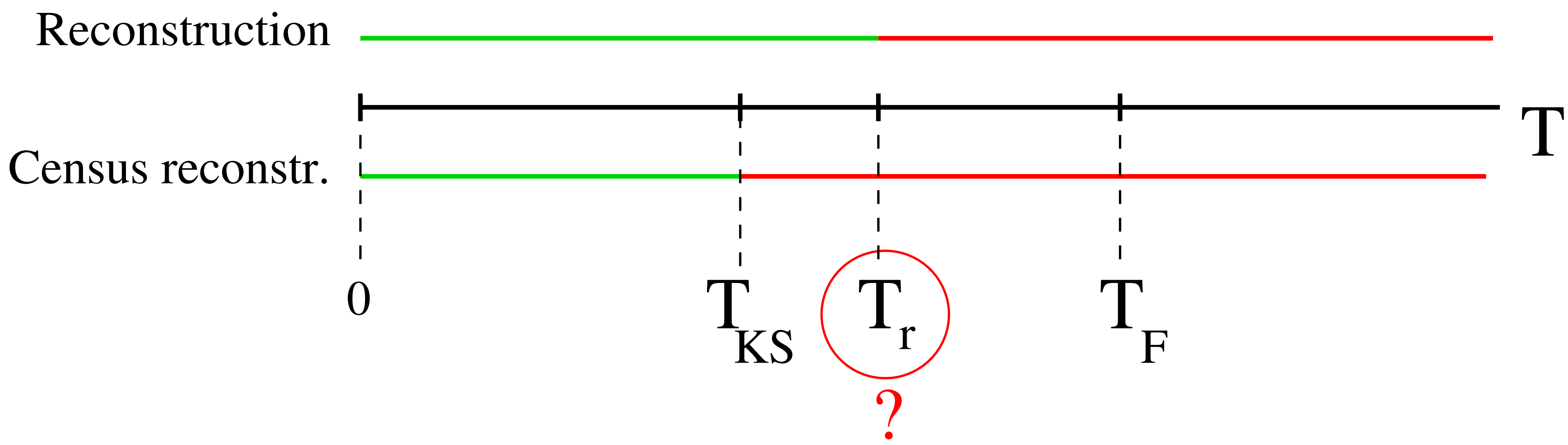
$$\mathbb{E}(x_n) \sim \frac{1}{q} + C \left| 1 - \varepsilon \frac{q}{q-1} \right|^n$$

$$\sqrt{\mathbb{E}({x_n}^2) - [\mathbb{E}(x_n)]^2} \sim C' k^{-n/2}$$

$\rightarrow$ Census reconstruction possible if $\varepsilon < \varepsilon_{KS} = \frac{q-1}{q} \frac{\sqrt{k}-1}{\sqrt{k}}$

Th (Mossel Peres): Threshold for census reconstruction is ε_{KS}

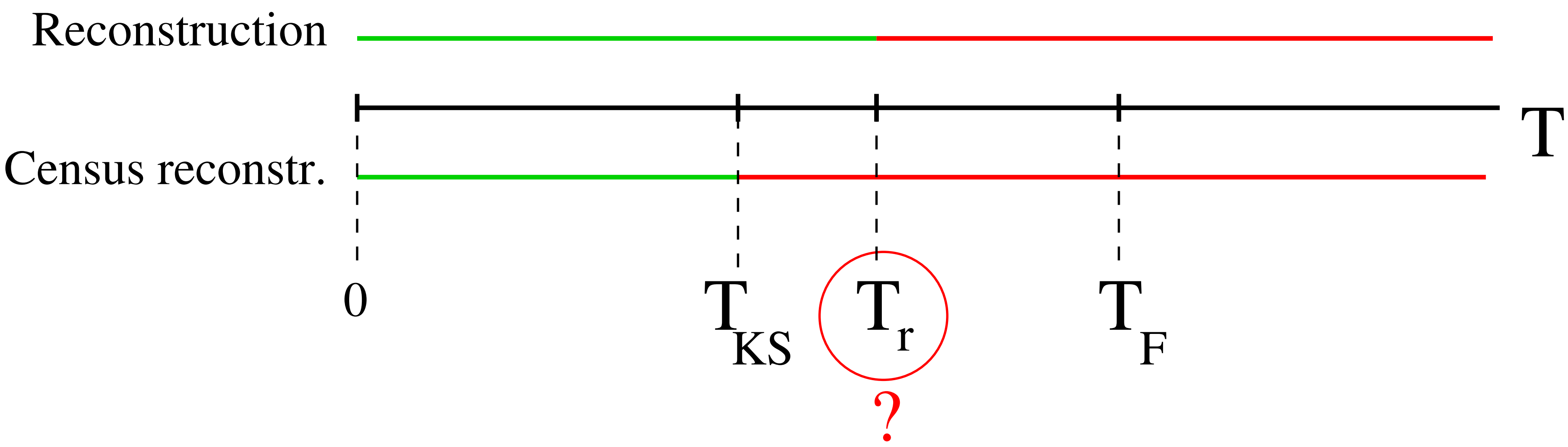
Some known results on the threshold T_r



T_{KS} given by: $\quad k\,|\lambda_2(\pi)|^2 = 1$, $\;T_F$ given by: $\quad k\,|\lambda_2(\pi)| = 1$

$T_{KS} \le T_r \le T_F$

Some known results on the threshold T_r



T_{KS} given by: $\quad k\,|\lambda_2(\pi)|^2 = 1$, $\;T_F$ given by: $\quad k\,|\lambda_2(\pi)| = 1$

$T_{KS} \leq T_r \leq T_F$

For $q = 2$: $T_r = T_{KS}$ (Bleher et al 95)

For q large enough: $T_r > T_{KS}$ (Mossel Peres 02)

New results (any tree, any channel)

- Reconstruction threshold T_r coincides with the dynamical (replica symmetry breaking) spin glass transition for an associated statistical physics problem

- Numerical procedure $\longrightarrow$ locate T_r with good precision

- Variational principle $\longrightarrow$ new rigorous bounds on T_r (proven for antiferromagnetic -or in general 'frustrated'- channels)

New results: examples

Ferromagnetic Potts

Numerically: $T_r = T_{KS}$ for $q = 3, 4$ and $k \in [2, 30]$

$T_r > T_{KS}$ for $q \geq 5$, $k \geq 2$

New results: examples

Ferromagnetic Potts

Numerically: $T_r = T_{KS}$ for $q = 3, 4$ and $k \in [2, 30]$

$T_r > T_{KS}$ for $q \geq 5$, $k \geq 2$

Antiferromagnetic Potts (coloring)

Numerically: Reconstruction in the noiseless limit (proper coloring) is possible only if $k \geq k_*(q)$, with $k_*(3) = 5$, $k_*(4) = 8$, $k_*(5) = 13, \dots$

New results: examples

Ferromagnetic Potts

Numerically: $T_r = T_{KS}$ for $q = 3, 4$ and $k \in [2, 30]$

$T_r > T_{KS}$ for $q \geq 5$, $k \geq 2$

Antiferromagnetic Potts (coloring)

Numerically: Reconstruction in the noiseless limit (proper coloring) is possible only if $k \geq k_*(q)$, with $k_*(3) = 5,\ k_*(4) = 8,\ k_*(5) = 13, \dots$

$T_r = T_{KS}$ for $q = 3$ and $k \in [5, 20]$

$T_r > T_{KS}$ for $q \geq 4$, $k \geq k_*(q)$.

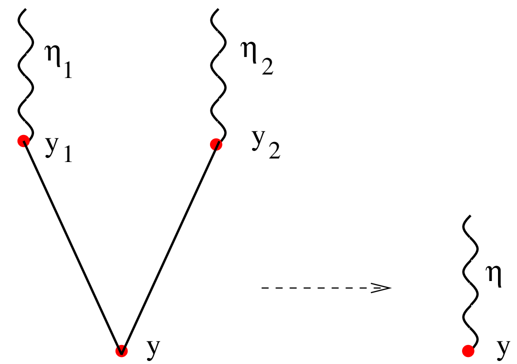
Rigorous: $k_*(4) \leq 8,\ k_*(5) \leq 13$. Discontinuous transition ($T_r > T_{KS}$) for $q = 4,\ k \in [9, 15]$, for $q = 5,\ k \in [13, 20]$, for $q = 6,\ k = 20$.

Reconstruction from a given boundary: recursion

Given a boundary:

$$\eta(y) = \frac{1}{z(\{\eta_i\})} \prod_{i=1}^{k} \left(\sum_{y_i=1}^{q} \pi(y_i|y)\, \eta_i(y_i) \right)$$

$$z(\{\eta_i\}) \equiv \sum_{y=1}^{q} \prod_{i=1}^{k} \left(\sum_{y_i} \pi(y_i|y)\, \eta_i(y_i) \right)$$



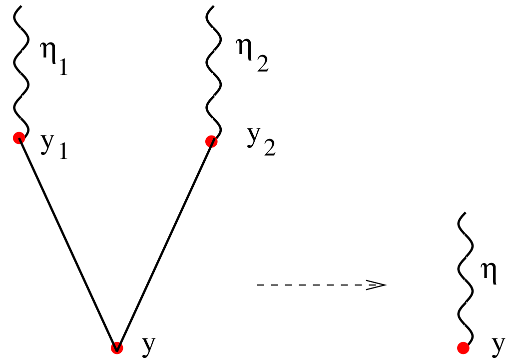
Mapping $\eta = \mathsf{F}(\eta_1, \ldots, \eta_k)$

Reconstruction from a given boundary: recursion

Given a boundary:

$$\eta(y) = \frac{1}{z(\{\eta_i\})} \prod_{i=1}^{k} \left(\sum_{y_i=1}^{q} \pi(y_i|y)\, \eta_i(y_i) \right)$$

$$z(\{\eta_i\}) \equiv \sum_{y=1}^{q} \prod_{i=1}^{k} \left(\sum_{y_i} \pi(y_i|y)\, \eta_i(y_i) \right)$$



Mapping $\eta = \mathsf{F}(\eta_1, \dots, \eta_k)$

Boundary B fixed by broadcast: $\eta_i(y_i) = \delta_{y_i, y_i^B}$ when i is a leaf.

Iterate from boundary to the center.

Statistics on the boundaries

For a given boundary B, on each site i of the tree, probability η_i^B, obtained by iteration from boundary to center.

NB: Link to Potts partition function $Z(y, B) = \sum_{\{y_i\}} \prod_{(ij)\in E} \pi(y_i, y_j)$:

Broadcast: $P_{\text{broadcast}}(B|y) = Z(y, B)$

Reconstruction: $\eta_i^B(y) = \frac{Z(y,B)}{\sum_{y'} Z(y',B)}$

Statistics on the boundaries

For a given boundary B, on each site i of the tree, probability η_i^B, obtained by iteration from boundary to center.

NB: Link to Potts partition function $Z(y,B) = \sum_{\{y_i\}} \prod_{(ij)\in E} \pi(y_i, y_j)$:

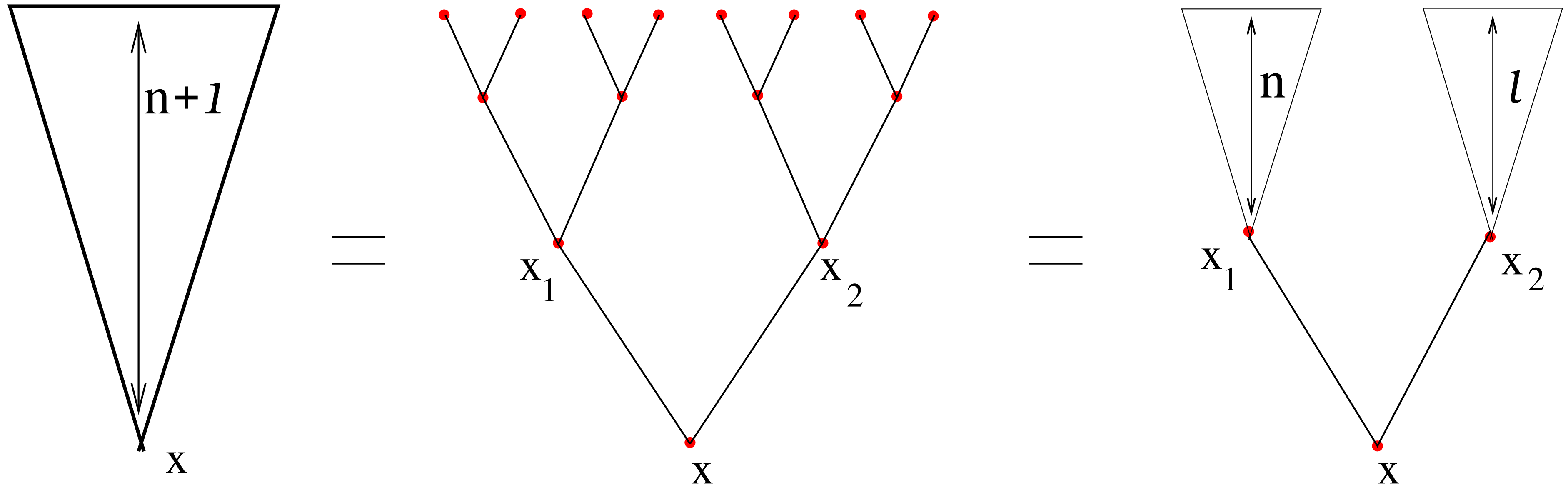
Broadcast: $P_{\text{broadcast}}(B|y) = Z(y,B)$

Reconstruction: $\eta_i^B(y) = \frac{Z(y,B)}{\sum_{y'} Z(y',B)}$

When B is generated randomly from broadcast (starting from a root fixed to x_0) $\to$ probability distribution $Q_{x_0}(\eta)$ of η on the root.

$$Q_{x_0}(\eta) = \sum_B Z(x_0, B) \prod_x \delta\left(\eta(x) - \frac{Z(x,B)}{\sum_{x'} Z(x',B)}\right)$$

Functional recursion



$$Q_x^{(n+1)}(\eta) = \sum_{x_1 \ldots x_k} \prod_{i=1}^{k} \pi(x_i|x) \int \delta\left[\eta - \mathsf{F}(\eta_1, \ldots, \eta_k)\right] \prod_{i=1}^{k} Q_{x_i}^{(n)}(\eta_i) d[\eta_i]$$

The spin glass fixed point

Symmetry property: $Q_x^{(n)}(\eta) = q\ \eta(x)\ \widehat{Q}^{(n)}(\eta)$ and $\widehat{Q}^{(n)}(\eta^\sigma) = \widehat{Q}^{(n)}(\eta)$

The spin glass fixed point

Symmetry property: $Q_x^{(n)}(\eta) = q\ \eta(x)\ \widehat{Q}^{(n)}(\eta)$ and $\widehat{Q}^{(n)}(\eta^\sigma) = \widehat{Q}^{(n)}(\eta)$

Recursion for $\widehat{Q}$:

$$\widehat{Q}^{(n+1)}(\eta) = q^{k-1} \int\ z(\{\eta_i\})\ \ \delta\left[\eta - \mathsf{F}(\eta_1, \ldots, \eta_k)\right]\ \prod_{i=1}^{k} \widehat{Q}^{(n)}(\eta_i)\ d[\eta_i]$$

The spin glass fixed point

Symmetry property: $Q_x^{(n)}(\eta) = q\;\eta(x)\;\widehat{Q}^{(n)}(\eta)$ and $\widehat{Q}^{(n)}(\eta^\sigma) = \widehat{Q}^{(n)}(\eta)$

Recursion for $\widehat{Q}$:

$$\widehat{Q}^{(n+1)}(\eta) = q^{k-1} \int \; z(\{\eta_i\}) \;\; \delta\left[\eta - \mathsf{F}(\eta_1,\ldots,\eta_k)\right] \; \prod_{i=1}^{k} \widehat{Q}^{(n)}(\eta_i)\; d[\eta_i]$$

Fixed point:

$$\widehat{Q}^{*}(\eta) = q^{k-1} \int \; z(\{\eta_i\}) \;\; \delta\left[\eta - \mathsf{F}(\eta_1,\ldots,\eta_k)\right] \; \prod_{i=1}^{k} \widehat{Q}^{*}(\eta_i)\; d[\eta_i]$$

The spin glass fixed point

Symmetry property: $Q_x^{(n)}(\eta) = q\ \eta(x)\ \widehat{Q}^{(n)}(\eta)$ and $\widehat{Q}^{(n)}(\eta^\sigma) = \widehat{Q}^{(n)}(\eta)$

Recursion for $\widehat{Q}$:

$$\widehat{Q}^{(n+1)}(\eta) = q^{k-1} \int\ z(\{\eta_i\})\ \ \delta\,[\eta - \mathsf{F}(\eta_1,\ldots,\eta_k)]\ \textstyle\prod_{i=1}^k \widehat{Q}^{(n)}(\eta_i)\ d[\eta_i]$$

Fixed point:

$$\widehat{Q}^{*}(\eta) = q^{k-1} \int\ z(\{\eta_i\})\ \ \delta\,[\eta - \mathsf{F}(\eta_1,\ldots,\eta_k)]\ \textstyle\prod_{i=1}^k \widehat{Q}^{*}(\eta_i)\ d[\eta_i]$$

Spin glass phase ("1-RSB"): exists iff there is a non-trivial *symmetric* fixed point.

Th: Reconstruction is possible iff there is a spin glass solution $\widehat{Q}^{*}$

Numerical approach

To obtain T_r: Solve the fixed point equation

$$\widehat{Q}^*(\eta) = q^{k-1} \int \; z(\{\eta_i\}) \;\; \delta\left[\eta - \mathsf{F}(\eta_1, \ldots, \eta_k)\right] \; \prod_{i=1}^{k} \widehat{Q}^*(\eta_i) \; d[\eta_i]$$

by a 'population dynamics' ($\sim$ Monte Carlo) method.

$Results$

Variational principle

"Complexity" of a distribution $\widehat{Q}$:

$$\Sigma(\widehat{Q}) = \frac{k+1}{2} \int \widehat{W}_{\rm e}(\eta_1, \eta_2) \;\; d\eta_1 \widehat{Q}(\eta_1)\, d\eta_2 \widehat{Q}(\eta_2)$$
$$- \int \widehat{W}_{\rm v}(\eta_1, \dots, \eta_{k+1}) \;\; \prod_{i=1}^{k+1} d\eta_i \widehat{Q}(\eta_i)$$

where $\widehat{W}_{\rm e}$ and $\widehat{W}_{\rm v}$ are known...

Theorem: A fixed point $\widehat{Q}^*$ is a stationary point of $\Sigma(\widehat{Q})$.

Conjecture: If there exists a symmetric distribution $\widehat{Q}$ such that $\Sigma(\widehat{Q}) > 0$, then the reconstruction problem is solvable.

Theorem: In the antiferromagnetic channel, if there exists a symmetric distribution $\widehat{Q}$ such that $\Sigma(\widehat{Q}) > 0$, then the reconstruction problem is solvable.

Practical use of the variational principle

Compute Σ within some restricted subspace. Define e.g. $\widehat{Q}_\mu$ which attributes equal weight $1/q$ to the q points $\eta = \gamma^{(x)}$, $x \in \{1, \dots, q\}$:

$$\gamma^{(x)}(y) = \begin{cases} 1-\mu & \text{if } y = x, \\ \mu/(q-1) & \text{otherwise.} \end{cases} \quad \text{and} \quad \Sigma(\mu) = \Sigma(\widehat{Q}_\mu).$$

Practical use of the variational principle

Compute Σ within some restricted subspace. Define e.g. $\widehat{Q}_\mu$ which attributes equal weight $1/q$ to the q points $\eta = \gamma^{(x)}$, $x \in \{1, \ldots, q\}$:

$$\gamma^{(x)}(y) = \begin{cases} 1-\mu & \text{if } y = x, \\ \mu/(q-1) & \text{otherwise.} \end{cases} \quad \text{and} \quad \Sigma(\mu) = \Sigma(\widehat{Q}_\mu).$$

SG

Practical use of the variational principle

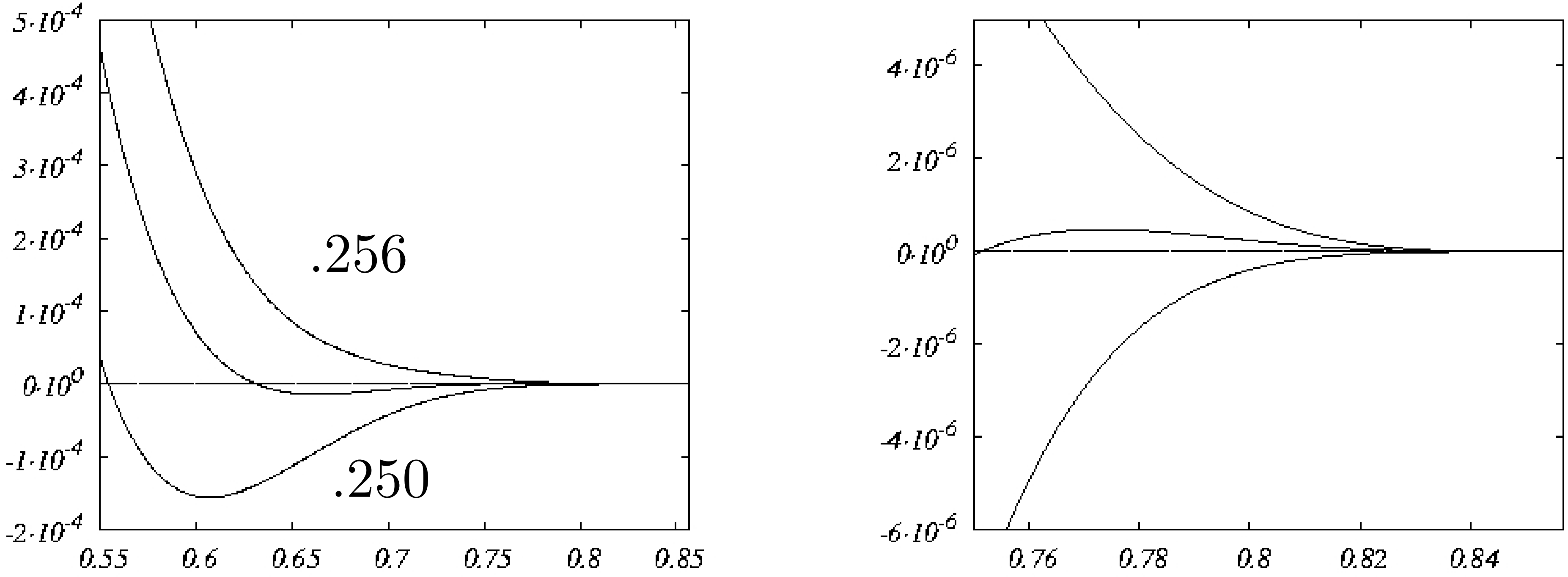
Compute Σ within some restricted subspace. Define e.g. $\widehat{Q}_\mu$ which attributes equal weight $1/q$ to the q points $\eta = \gamma^{(x)}$, $x \in \{1,\dots,q\}$:

$$\gamma^{(x)}(y) = \begin{cases} 1-\mu & \text{if } y = x, \\ \mu/(q-1) & \text{otherwise.} \end{cases} \quad \text{and} \quad \Sigma(\mu) = \Sigma(\widehat{Q}_\mu).$$

SG

Example: ferromagnetic Potts, $k=2,\ q=7$

Ferromagnetic Potts, $k = 2,\ q = 7$: plot of $-\Sigma$ vs μ:



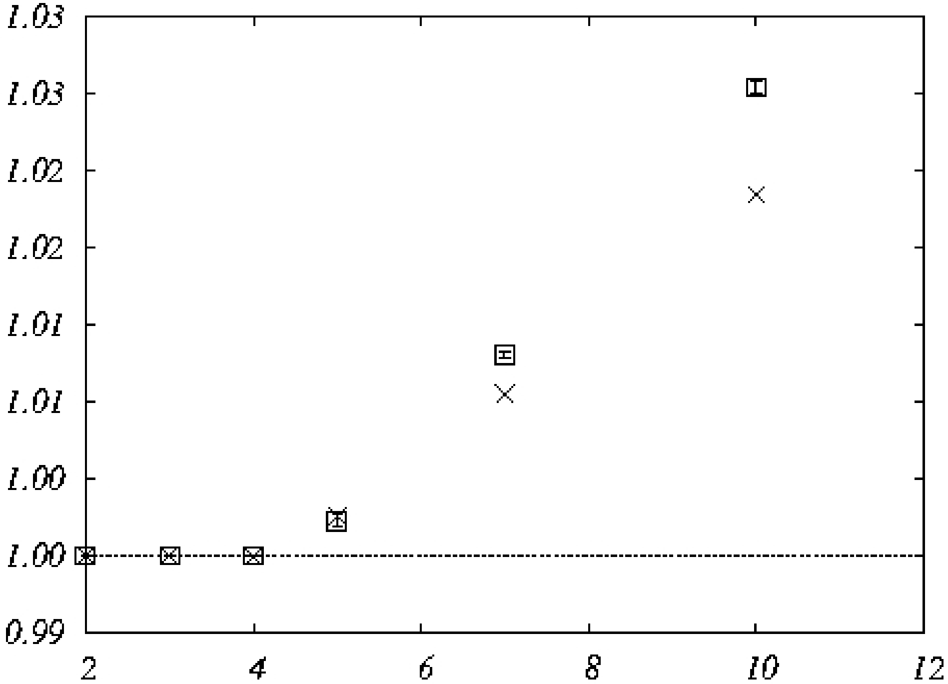
$k = 2,\ q = 7.\ \varepsilon = 0.250,\ 0.253,\ 0.256.$

First order transition: $\varepsilon_{\mathrm{KS}}$ found by $\frac{d\Sigma}{d\mu}(\mu = (q-1)/q) = 0$

$\varepsilon_{\mathrm{KS}} = 0.2510513..;\ \varepsilon_{\mathsf{var}} = .25369...;\ \varepsilon_r \simeq .25432$

Results for the ferromagnetic Potts channel

$\varepsilon_{\rm r}/\varepsilon_{\rm KS}$ as a function of q, for $k = 2$



Squares: $\varepsilon_{\rm r}(k, q)$. Crosses: variational lower bound.

Spin glass theory on a tree 1

Broadcast: generates an equilibrium configuration of the Potts model with free boundary conditions.

Reconstruction: given the boundary B obtained from the broadcast, the conditional probability of the variable on the root, $P(x|B)$, is also given by Boltzmann's measure for the Potts model. But B creates some frustration

Spin glass theory on a tree 1

Broadcast: generates an equilibrium configuration of the Potts model with free boundary conditions.

Reconstruction: given the boundary B obtained from the broadcast, the conditional probability of the variable on the root, $P(x|B)$, is also given by Boltzmann's measure for the Potts model. But B creates some frustration

Spin glass on a tree: frustration only through boundary conditions.

Simple Ising spin glass model (Chayes, Chayes, Sethna, Thouless 1986): fix each spin on the boundary to ± 1 with probability $1/2$. But no RSB, no real spin glass phase.

Spin glass theory on a tree 2

Other boundary condition: $\prod_{i \in \{\text{ leaves}\}} \eta_i(x_i)$, but with correlated η_i:

$$\mathbb{P}(\{\eta_i\}) = \frac{1}{\Xi_L} Z_L(\{\eta_i\}) \prod_{i \in \text{leaves}} \widetilde{Q}^{(0)}(\eta_i) \ ,$$

where $\widetilde{Q}^{(0)}(\eta)$ is the uniform distribution on the q 'corners' of the simplex $\eta(x) = \delta_{x,r}, \quad r \in \{1, \dots, q\}$

Spin glass theory on a tree 2

Other boundary condition: $\prod_{i\in\{\text{ leaves}\}} \eta_i(x_i)$, but with correlated η_i:

$$\mathbb{P}(\{\eta_i\}) = \frac{1}{\Xi_L} Z_L(\{\eta_i\}) \prod_{i\in\text{leaves}} \widetilde{Q}^{(0)}(\eta_i) \ ,$$

where $\widetilde{Q}^{(0)}(\eta)$ is the uniform distribution on the q 'corners' of the simplex $\eta(x) = \delta_{x,r}, \quad r \in \{1,\dots,q\}$

$\longrightarrow$ functional recursion: identical to the one found in reconstruction

If $\widetilde{Q}^{(0)} = \widehat{Q}^*$, this model is statistically invariant by translation (provided rooted tree $\rightarrow$ regular Cayley tree): The properties of a spin don't depend on its shell.

Spin glass theory: Bethe lattice

Traditionally, "Bethe lattice" $=$ interior of a Cayley tree

Frustrated systems: frustration from the boundary $\longrightarrow$ bad definition.

Spin glass theory: Bethe lattice

Traditionally, "Bethe lattice" $=$ interior of a Cayley tree

Frustrated systems: frustration from the boundary $\rightarrow$ bad definition.

Better definition (M$+$Parisi 2001): use a random regular graph with fixed degree $k+1$ on each vertex.

Local structure (from a generic point, to any finite depth) $=$ tree.

Frustration from long loops (size of $O(\log N)$).

This work: $\rightarrow$ Typical boundary condition from outside the tree $=$ the one obtained by broadcast !

Cavity method

Analysis of Potts model on a random regular graph: cavity method $\rightarrow$ iterative functional equations.

$\eta_{i\rightarrow j}(x_i)$ = marginal distribution of x_i when the edge $i - j$ has been cut = function of the distributions $\eta_{l\rightarrow i}(x_l)$ where l are the neighbors of i different from j.

Cavity method

Analysis of Potts model on a random regular graph: cavity method $\rightarrow$ iterative functional equations.

$\eta_{i\rightarrow j}(x_i)$ = marginal distribution of x_i when the edge $i-j$ has been cut = function of the distributions $\eta_{l\rightarrow i}(x_l)$ where l are the neighbors of i different from j.

'Liquid' or 'paramagnetic' solution, uniform: $\eta_{i\rightarrow j}(x_i) = \eta(x_i)$

Spin glass: many modulated solutions: $\eta^{\alpha}_{i\rightarrow j}(x_i)$. Functional $\widehat{Q}^*(\eta)$= probability that $\eta^{\alpha}_{i\rightarrow j} = \eta$, when α is chosen randomly with its Boltzmann weight. $e^{N\Sigma}$ is the number of modulated solutions (BP fixed points)

Cavity method

Analysis of Potts model on a random regular graph: cavity method $\rightarrow$ iterative functional equations.

$\eta_{i\to j}(x_i)$ = marginal distribution of x_i when the edge $i - j$ has been cut = function of the distributions $\eta_{l\to i}(x_l)$ where l are the neighbors of i different from j.

'Liquid' or 'paramagnetic' solution, uniform: $\eta_{i\to j}(x_i) = \eta(x_i)$

Spin glass: many modulated solutions: $\eta^{\alpha}_{i\to j}(x_i)$. Functional $\widehat{Q}^*(\eta)$= probability that $\eta^{\alpha}_{i\to j} = \eta$, when α is chosen randomly with its Boltzmann weight. $e^{N\Sigma}$ is the number of modulated solutions (BP fixed points)

(NB: spin glass phase may be hidden by a ferromagnetic state, if it exists)

Comments

A very interesting problem!

Deep connexions to spin glasses

Using spin glass methods: $\longrightarrow$ new exact results (for frustrated case) and conjectures

Several open questions: prove variational conjecture also in unfrustrated cases $\longrightarrow$ best known bounds... Meaning of the complexity directly in the broadcast/reconstruction problem?

Comments

A very interesting problem!

Deep connexions to spin glasses

Using spin glass methods: $\longrightarrow$ new exact results (for frustrated case) and conjectures

Several open questions: prove variational conjecture also in unfrustrated cases $\longrightarrow$ best known bounds... Meaning of the complexity directly in the broadcast/reconstruction problem?

Ref: "Reconstruction on trees and spin glass transition", Marc Mézard and Andrea Montanari, J. Stat. Phys. 124 (2006) 1317-1350

Appendix A: Proof (sketch)

Proposition: The reconstruction problem is solvable iff there is a non-trivial fixed point $\widehat{Q}^*(\eta)$

If reconstruction solvable: Sequence of $\widehat{Q}^{(n)}$ converges weakly to $\widehat{Q}^*(\eta)$ which is non-trivial.

If $\widehat{Q}^*$ exists, non-trivial. Construct the q probabilities $Q^*_x(\eta) = q\ \eta(x)\ \widehat{Q}^*(\eta)$. Use them to infer some information on the root. On a leaf i, broadcast has generated symbol x_i. Generate η_i from $Q^*_{x_i}$. Given the η's in generation n: generate the new η's in generation $n-1$ from the mapping $\eta = \mathsf{F}(\eta_1, \ldots, \eta_k)$, down to the root. For each site j, conditional to the broadcast having produced $X_j = x_j$, the η_j provided by the above procedure is distributed according to $Q^*_{x_j}$ (Thanks to James Martin)

Appendix B: Variational principle 1

"Complexity" of a distribution $\widehat{Q}$:

$$\Sigma(\widehat{Q}) = \frac{k+1}{2}\int \widehat{W}_{\mathrm{e}}(\eta_1,\eta_2)\;\; d\eta_1\widehat{Q}(\eta_1)\, d\eta_2\widehat{Q}(\eta_2)$$

$$\int \widehat{W}_{\mathrm{v}}(\eta_1,\dots,\eta_{k+1})\;\; \prod_{i=1}^{k+1} d\eta_i\widehat{Q}(\eta_i)$$

where

$$\widehat{W}_{\mathrm{e}} \equiv -\left[\frac{\sum_{x_1,x_2}\eta_1(x_1)\eta_2(x_2)\pi(x_1,x_2)}{\sum_{x_1,x_2}\overline{\eta}(x_1)\overline{\eta}(x_2)\pi(x_1,x_2)}\right]\log\left[\frac{\sum_{x_1,x_2}\eta(x_1)\eta(x_2)\pi(x_1,x_2)}{\sum_{x_1,x_2}\overline{\eta}(x_1)\overline{\eta}(x_2)\pi(x_1,x_2)}\right]$$

$$\widehat{W}_{\mathrm{v}} \equiv -\left[\frac{\sum_x\prod_i\sum_{x_i}\eta_i(x_i)\pi(x,x_i)}{\sum_x\prod_i\sum_{x_i}\overline{\eta}(x_i)\pi(x,x_i)}\right]\log\left[\frac{\sum_x\prod_i\sum_{x_i}\eta_i(x_i)\pi(x,x_i)}{\sum_x\prod_i\sum_{x_i}\overline{\eta}(x_i)\pi(x,x_i)}\right]\,.$$

$$\overline{\eta}(x) = 1/q$$

Variational principle 2

Proposition: A fixed point $\widehat{Q}^*$ is a stationary point of $\Sigma(Q)$.

(Precisely: given any symmetric distribution $\widehat{Q}$, define

$\Sigma^*(t) \equiv \Sigma[(1-t)\widehat{Q}^* + t\widehat{Q}]$. Then $\left.\frac{d\Sigma^*}{dt}\right|_{t=0} = 0$).

Proposition In the antiferromagnetic Potts channel, if there exists a symmetric distribution $\widehat{Q}$ such that $\Sigma(Q) < 0$, then the reconstruction problem is solvable.

Conjecture In any channel, if there exists a symmetric distribution $\widehat{Q}$ such that $\Sigma(Q) < 0$, then the reconstruction problem is solvable.

q	k	ε_{r}	$\varepsilon_{\mathrm{KS}}$
5	2	0.2348(1)	0.2343146
5	3	0.33881(5)	0.3381198
5	4	0.4008(1)	0.4
5	7	0.4986(1)	0.4976284
5	15	0.5955(1)	0.5934409
7	2	0.25432(5)	0.2510513
7	4	0.43325(5)	0.4285714
10	2	0.2716(2)	0.2636039
15	2	0.2881(1)	0.2733670

Table 1: Threshold for the ferromagnetic Potts channel

q	k	$\varepsilon_{\rm r}$	$\varepsilon_{\rm KS}$	$\varepsilon_{\rm var}$	$\varepsilon_{\rm alg}$	$\varepsilon_{\rm MP}$	I_*	
5	2	0.2348(1)	0.2343146	0.23491	$---$	0.30264	0.052(5)	0
5	3	0.33881(5)	0.3381198	0.33887	0.19047	0.41712	0.06(2)	
5	4	0.4008(1)	0.4	0.40081	0.29046	0.48	0.06(1)	
5	7	0.4986(1)	0.4976284	0.49847	0.41114	0.57143	0.07(1)	
5	15	0.5955(1)	0.5934409	0.59422	0.53965	0.65238	0.14(1)	
7	2	0.25432(5)	0.2510513	0.25369	$---$	0.34577	0.14(1)	
7	4	0.43325(5)	0.4285714	0.43250	0.30769	0.53909	0.195(5)	
10	2	0.2716(2)	0.2636039	0.26977	$---$	0.38325	0.23(2)	
15	2	0.2881(1)	0.2733670	0.28472	$---$	0.41652	0.37(3)	

Table 2: Thresholds (numerical results and bounds) for the ferromagnetic Potts channel. The reconstruction threshold $\varepsilon_{\rm r}$ satisfies the rigorous bounds $\varepsilon_{\rm r} \geq \varepsilon_{\rm KS}$, $\varepsilon_{\rm r} \geq \varepsilon_{\rm alg}$, and $\varepsilon_{\rm r} \leq \varepsilon_{\rm MP}^{-}$. The conjectured variational principle would imply $\varepsilon_{\rm r} \geq \varepsilon_{\rm var}$.

q	k	ε_{r}	$\varepsilon_{\mathrm{KS}}$	$\varepsilon_{\mathrm{var}}$	$\varepsilon_{\mathrm{alg}}$	$\varepsilon^{-}_{\mathrm{MP}}$	I_*	
4	8	0.99953(4)	−−	−−	−−	0.91552	1.56(4)	0
4	9	0.9908(4)	1	0.99298	−−	0.90717	1.31(2)	0
4	10	0.9820(8)	0.9871708	0.98304	−−	0.9	1.2(2)	0
4	11	0.9725(3)	0.9761335	0.97363	0.99736	0.89376	1.07(5)	0
4	12	0.9643(3)	0.9665063	0.96498	0.98946	0.88826	0.26(3)	4
4	15	0.9431(3)	0.9436492	0.94338	0.96903	0.875	0.5(1)	0
4	18	0.9267(2)	0.9267766	0.92686	0.95264	0.86502	0.3(1)	0
5	13	0.99741(5)	−−	0.99982	−−	0.92308	1.76(4)	0
5	14	0.9932(1)	−−	0.99555	−−	0.91916	1.7(1)	0
5	15	0.9888(1)	−−	0.99092	−−	0.91561	1.48(5)	0
5	20	0.9685(3)	0.9788854	0.96991	0.98581	0.90177	1.1(5)	0
6	17	0.999924(5)	−−	−−	−−	0.93482	2.20(4)	0.6
6	20	0.9932(3)	−−	0.99546	−−	0.92792	1.87(6)	0.5

Table 3: Antiferromagnetic, rigorous bounds: $\varepsilon_{\mathrm{r}} \le \varepsilon_{\mathrm{KS}}$ (KS), $\varepsilon_{\mathrm{r}} \le \varepsilon_{\mathrm{alg}}$ (Mossel), $\varepsilon_{\mathrm{r}} \le \varepsilon_{\mathrm{var}}$ (M+M), $\varepsilon_{\mathrm{r}} \ge \varepsilon^{-}_{\mathrm{MP}}$ (Mossel Peres).